POISSON STRUCTURES

1. Historical background

Let us recall the *Newton's second law*:

$$Mass \times Acceleration = Force.$$

The force is described by a potential field particle $V : \mathbb{R}^3 \to \mathbb{R}$. Concretely, $Force = -\nabla V$.

Let us denote by m the mass of the particle and by $q = (q_1, q_2, q_3) \in \mathbb{R}^3$ its coordinates position in the space. Then the equation of motion can be written as

$$m\,\ddot{q} = -\nabla V.$$

This is a second order differential equation in $\mathbb{R}^3$ (here, $\mathbb{R}^3$ is the *configuration space*).
In each coordinate:

$$m\,\ddot{q}_i = -\frac{\partial V}{\partial q_i}, \qquad i = 1, 2, 3.$$

$$\dot{q} : \text{velocity of the particle,}$$

$$p := m\,\dot{q} : \text{momentum of the particle.}$$

What Newton really stated:

$$Force = \text{change in momentum.}$$

Energy of the particle:

$$\begin{aligned} H &= \text{Kinetic energy} + \text{Potential energy} \\ &= \frac{1}{2}\,m \sum_{i=1}^{3} q_i^2 + V(q) \\ &= \frac{1}{2m} \sum_{i=1}^{3} p_i^2 + V(q). \end{aligned}$$

Another way to write the equations of motion:

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}, \\ \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}, \end{cases}$$

which is a system of first order differential equations on $\mathbb{R}^6$ (here, $\mathbb{R}^6$ is the *phase space*). This is the same as giving a vector field in $\mathbb{R}^6$, hence we have two points of view for the solutions of this system:

(i) as the time evolution of the system, and
(ii) as integral curves of a vector field.

Let us notice that H is a *constant of motion* (*i.e.*, $\dot{H} = 0$). In effect,

$$\begin{aligned}\dot{H} &= \sum_{i=1}^{3} \left(\frac{\partial H}{\partial q_i}\, \dot{q}_i + \frac{\partial H}{\partial p_i}\, p_i \right) \\ &= \sum_{i=1}^{3} \left(\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0.\end{aligned}$$

For any function $f : \mathbb{R}^6 \to \mathbb{R}$, what is $\dot{f}$?

$$\dot{f} = \underbrace{\sum_{i=1}^{3} \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right)}_{\text{Poisson bracket between } f \text{ and } H}.$$

For any pair of functions $f, g : \mathbb{R}^6 \to \mathbb{R}$,

$$\{f, g\} := \sum_{i=1}^{3} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

In these terms the equations of motion can be written as follows:

$$\begin{cases} \dot{q}_i = \{q_i, H\}, \\ \dot{p}_i = \{p_i, H\}. \end{cases}$$

The property of H being a constant of motion for the mechanical system is an algebraic manifestation of the skew-symmetry of the Poisson bracket:

$$\{H, H\} = \dot{H} = 0.$$

In fact,

$$\begin{aligned} f \text{ is a constant of motion} &\Leftrightarrow \dot{f} = 0 \\ &\Leftrightarrow \{f, H\} = 0. \end{aligned}$$

Constants of motion are important!

Theorem 1.1 (Poisson's theorem)**.** *If f and g are constants of motion then $\{f, g\}$ is also a constant of motion.*

In formulas:

$$\left. \begin{aligned} \{f, H\} = 0 \\ \{g, H\} = 0 \end{aligned} \right\} \Rightarrow \{\{f, g\}, H\} = 0.$$

Jacobi, 30 years later:

$$\{\{f, g\}, H\} + \{\{H, f\}, g\} + \{\{g, H\}, f\} = 0. \qquad \textit{Jacobi identity}$$

What a Poisson bracket would be?

$\{\cdot, \cdot\}$ is skew-symmetric,
$\{\cdot, \cdot\}$ satisfies the Jacobi identity,
$\{\cdot, \cdot\}$ should be a derivation (like a vector field).

2. A brief introduction to Poisson brackets

We are thinking of a Poisson brackt as a map $\{\cdot,\cdot\} : \mathcal{A}\times\mathcal{A}\to\mathcal{A}$, where $\mathcal{A}$ is some *nice* space. The most reasonable thing to expect is that $\mathcal{A}$ be an *associative commutative algebra.*

Let us fix a ground field $\mathbb{F}$ of characteristic zero (think for example on $\mathbb{R}$ or $\mathbb{C}$).

Definition 2.1. An algebra over $\mathbb{F}$ is an $\mathbb{F}$-vector space $\mathcal{A}$ with a bilinear map $\mu : \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ (*i.e.*, a product). It is said to be

commutative if for every $a, b \in \mathcal{A}$, $\mu(b, a) = \mu(a, b)$;
associative if for every $a, b, c \in \mathcal{A}$, $\mu(\mu(a, b), c) = \mu(a, \mu(b, c))$;
unitary if there exists $e \in \mathcal{A}$ such that for every $a \in \mathcal{A}$, $\mu(a, e) = \mu(e, a) = a$.

Example 2.2.

1) $\mathbb{F}[x_1, \dots, x_n]$, polynomials in n variables (n could be ∞). Here, μ is the usual product of polynomials. It is commutative, associative and has a unit.
2) The $n\times n$ matrices with matrix multiplication. It is associative, has a unit but it is not commutative.
3) Functions on $\mathbb{R}^n$ or $\mathbb{C}^n$ (continuous, $\mathcal{C}^\infty$, polynomial, holomorphic, etc.).
4) Functions on a sphere, $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, or on any variety.

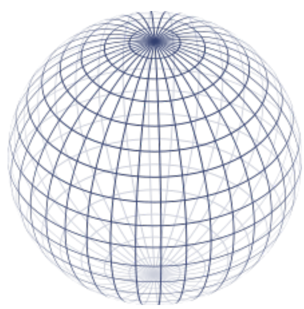


Figure 1. 2-sphere.

Some additional definitions:

- Let $\mathcal{A}$ be an associative commutative algebra over $\mathbb{F}$. A subset $I \subseteq \mathcal{A}$ is said to be an ideal of $\mathcal{A}$ if for every $a \in \mathcal{A}$ and $x \in \mathcal{I}$, $x\,a \equiv x\cdot a \equiv \mu(x, a) \in I$ (otherwise written, $I\cdot A \subseteq I$). I is proper if I is different from $\{0\}$ and $\mathcal{A}$.
- Let $\mathcal{A}, \mathcal{B}$ be two commutative associative algebras. A linear map $\varphi : \mathcal{A}\to\mathcal{B}$ is said to be a morphism of algebras if $\varphi(a\cdot a') = \varphi(a)\cdot\varphi(a')$.

Proposition 2.3.

- If I is a proper ideal of $\mathcal{A}$ then $\mathcal{A}/I$ has a unique algebra structure such that the canonical projection $\pi : \mathcal{A}\to\mathcal{A}/I$ is a morphism of algebras.

◇ If $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism of algebras then $\operatorname{Ker}(\varphi) := \{a \in \mathcal{A} \mid \varphi(a) = 0\}$ is an ideal of $\mathcal{A}$ and φ factors as

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\ \downarrow{\scriptstyle \pi} & & \uparrow \\ \mathcal{A}/\operatorname{Ker}(\varphi) & \xrightarrow{\sim} & \operatorname{Im}(\varphi) \end{array}$$

where $\operatorname{Im}(\varphi) := \{b \in \mathcal{B} \mid \exists a \in \mathcal{A}, \varphi(a) = b\}$.

Example 2.4.

5) Let $f_1, \ldots, f_k \in \mathcal{A} = \mathbb{F}[x_1, \ldots, x_n]$. The ideal I generated by $f_1, \ldots, f_k$ (the smallest one containing each f_j) has the form $f_1\mathcal{A} + \cdots + f_k\mathcal{A}$. This ideal is denoted by $\langle f_1, \ldots, f_k\rangle$. The quotient algebra

$$\frac{\mathcal{A}}{I} = \frac{\mathbb{F}[x_1, \ldots, x_n]}{\langle f_1, \ldots, f_k\rangle}$$

can be viewed as functions on $V(f_1, \ldots, f_k) = \{x \in \mathbb{F}^n \mid \forall i = 1, \ldots, k, \quad f_i(x) = 0\}$.

If $f \in \mathcal{A}$, $\bar{f} = f + I \in \mathcal{A}/I$.

If $x \in V(f_1, \ldots, f_k)$ then $\bar{f}(x) = f(x)$, because $\lambda(x) = 0$ for every $\lambda \in I$.

Definition 2.5. An algebra $(\mathcal{A}, \mu)$ is said to be a Lie algebra if

μ is skew-symmetric: for all $a, b \in \mathcal{A}$, $\mu(b, a) = -\mu(a, b)$,

μ satisfies the Jacobi identity: for all $a, b, c \in \mathcal{A}$, $\mu(\mu(a, b), c) + \operatorname{cyclic}(a, b, c) = 0$.

Here, $\operatorname{cyclic}(a, b, c)$ indicates that the sum is performed over all cyclic permutations, that is,

$$\mu(\mu(a, b), c) + \mu(\mu(c, a), b) + \mu(\mu(b, c), a).$$

It is usual to denote μ by $[\cdot, \cdot]$ or $\{\cdot, \cdot\}$.

A subspace $\mathcal{B} \subseteq \mathcal{A}$ is said to be a

Lie subalgebra if $[\mathcal{B}, \mathcal{B}] \subseteq \mathcal{B}$,

Lie ideal if $[\mathcal{B}, \mathcal{A}] \subseteq \mathcal{B}$. In addition, $\mathcal{B}$ is proper if it is different from $\{0\}$ and $\mathcal{A}$.

Proposition 2.3 holds in the context of Lie algebras.

Example 2.6.

1) The space of $n \times n$ matrices with a new product, the commutator:

$$[A, B] = AB - BA.$$

Jacobi identity:

$$[[A, B], C] = ABC - BAC - CAB + CBA,$$
$$[[B, C], A] = BCA - CBA - ABC + ACB,$$
$$[[C, A], B] = CAB - ACB - BCA + BAC.$$

Hence, $[[A,B],C] + \mathrm{cyclic}(A,B,C) = 0$.

2) More generally, for any associative algebra the commutator is a Lie bracket.
3) $\mathbb{F}^3$ with the vector product:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

4) The $n \times n$ matrices of zero trace form a Lie subalgebra of the Lie algebra in the first example, because

$$\mathrm{tr}([A,B]) = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB - AB) = 0.$$

5) Upper triangular $n \times n$ matrices form a Lie subalgebra, hence a Lie subalgebra.
6) In the same way, skew-symmetric matrices ($A^T = -A$). In effect,

$$\begin{aligned} [A,B]^T &= (AB - BA)^T = (AB)^T - (BA)^T \\ &= B^T A^T - A^T B^T = (-B)(-A) - (-A)(-B) \\ &= BA - AB = -(AB - BA) \\ &= -[A,B]. \end{aligned}$$

Derivations and biderivations

Definition 2.7. Let $\mathcal{A}$ be an associative commutative algebra. A derivation of $\mathcal{A}$ (with values in $\mathcal{A}$) is a linear map $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\delta(ab) = \delta(a)b + a\delta(b).$$

Example 2.8.

1) Let $\varphi \in \mathbb{F}[x]$ be any polynomial. The map

$$\begin{array}{rccc} \mathcal{D} & : & \mathbb{F}[x] & \longrightarrow & \mathbb{F}[x] \\ & & f & \mapsto & \varphi f', \end{array}$$

is a derivation of $\mathbb{F}[x]$.

2) In the same way, let $\varphi \in \mathbb{F}[x_1; \ldots, x_n]$ be any polynomial. Then for all $i = 1, \ldots, n$, the map

$$\begin{array}{rccc} \dfrac{\partial}{\partial x_i} & : & \mathbb{F}[x_1, \ldots, x_n] & \longrightarrow & \mathbb{F}[x_1, \ldots, x_n] \\ & & f & \mapsto & \varphi \dfrac{\partial f}{\partial x_i}, \end{array}$$

is a derivation of $\mathbb{F}[x_1, \ldots, x_n]$.

3) All the derivations of $\mathbb{F}[x_1, \ldots, x_n]$ are of the form

$$\begin{array}{rccc} \displaystyle\sum_{i=1}^n g_i \frac{\partial}{\partial x_i} & : & \mathbb{F}[x_1, \ldots, x_n] & \longrightarrow & \mathbb{F}[x_1, \ldots, x_n] \\ & & f & \mapsto & \displaystyle\sum_{i=1}^n g_i \frac{\partial f}{\partial x_i}, \end{array}$$

for some $g_1, \ldots, g_n \in \mathbb{F}[x_1, \ldots, x_n]$.

Reason:

Every derivation $\mathcal{D}$ of $\mathbb{F}[x_1, \ldots, x_n]$ is determined by its values on $x_1, \ldots, x_n$. In fact, for $f \in \mathbb{F}[x_1, \ldots, x_n]$,

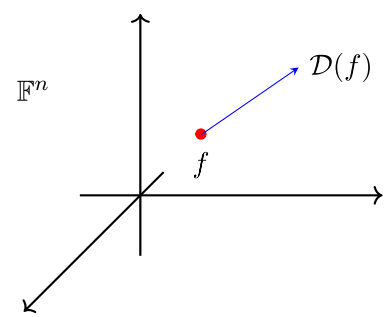
$$\mathcal{D}(f) = \sum_{i=1}^{n} \mathcal{D}(x_i) \frac{\partial f}{\partial x_i}.$$

Conversely, for any $g_1, \ldots, g_n \in \mathbb{F}[x_1, \ldots, x_n]$ the linear map defined by

$$\mathcal{D}(f) := \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_i}$$

is a derivation of $\mathbb{F}[x_1, \ldots, x_n]$.

Geometrically:



Remark 2.9.

- On a smooth manifold M there is a one to one correspondence between vector fields and derivations of $\mathcal{C}^\infty(M)$. This is not true for the complex case.
- For a general associative commutative algebra $\mathcal{A}$ every derivation is determined by its values on a set of generators. However the converse is not true.

  For example, on the sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, every derivation $\mathcal{D}$ must satisfy

$$x\mathcal{D}(x) + y\mathcal{D}(y) + z\mathcal{D}(z) = 0.$$

Proposition 2.10. The derivations of an associative commutative algebra $\mathcal{A}$ forms a Lie algebra under the commutator.

Proof.

We need to show that if δ and ρ are derivations of $\mathcal{A}$ then $[\delta, \rho] = \delta \circ \rho - \rho \circ \delta$ is also a derivation. In effect, for $a, b \in \mathcal{A}$,

$$\begin{aligned}
[\delta, \rho](ab) &= \delta \circ \rho(ab) - \rho \circ \delta(ab) \\
&= \delta(\rho(a)b + a\rho(b)) - \rho(\delta(a)b + a\delta(b)) \\
&= \delta \circ \rho(a)b + \rho(a)\delta(b) + \delta(a)\rho(b) + a\delta \circ \rho(b) \\
&\quad - \rho \circ \delta(a)b - \delta(a)\rho(b) - \rho(a)\delta(b) - a\rho \circ \delta(b) \\
&= (\delta \circ \rho(a) - \rho \circ \delta(a))\, b + a\,(\delta \circ \rho(b) - \rho \circ \delta(b)) \\
&= [\delta, \rho](a)\, b + a\, [\delta, \rho](b).
\end{aligned}$$

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Definition 2.11. A biderivation P of an associative commutative algebra $\mathcal{A}$ is a bilinear map $P : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying: for any $a \in \mathcal{A}$ the linear maps

$$\begin{array}{rcccl} P_a^1 & : & \mathcal{A} & \longrightarrow & \mathcal{A} \\ & & b & \mapsto & P(a,b) \end{array} \quad , \quad \begin{array}{rcccl} P_a^2 & : & \mathcal{A} & \longrightarrow & \mathcal{A} \\ & & b & \mapsto & P(b,a) \end{array}$$

are derivations of $\mathcal{A}$.

Example 2.12. On $\mathbb{F}[x, y]$ the bilinear map

$$\begin{array}{rcccl} P & : & \mathbb{F}[x,y] & \longrightarrow & \mathbb{F}[x,y] \\ & & (f,g) & \mapsto & \varphi \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} \end{array}$$

is a biderivation, for any polynomial $\varphi \in \mathbb{F}[x, y]$.

Proposition 2.13. Every skew-symmetric biderivation P of $\mathbb{F}[x_1, \ldots, x_n]$ is determined by its values $P(x_i, x_j)$ with $1 \leq i < j \leq n$. In fact,

$$P(f,g) = \sum_{1 \leq i < j \leq n} P(x_i, x_j) \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right).$$

In short,

$$P = \sum_{1 \leq i < j \leq n} P(x_i, x_j) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Remark 2.14. Geometrically, in the case of smooth manifolds skew-symmetric biderivations correspond to bivector fields.

In the same way one defines multiderivations.

Definition 2.15. A Poisson algebra $(\mathcal{A}, \cdot, \{\cdot, \cdot\})$ is the data of

$\mathcal{A}$, an $\mathbb{F}$-vector space,
$(\mathcal{A}, \cdot)$, an associative commutative algebra with unit,
$(\mathcal{A}, \{\cdot, \cdot\})$, a Lie algebra,
$\{\cdot, \cdot\}$, a biderivation.

Important feature:

For any $a \in \mathcal{A}$, $X_a := \{\cdot, a\} : \mathcal{A} \to \mathcal{A}$, $b \mapsto \{b, a\}$ is a derivation of $\mathcal{A}$, called the Hamiltonian derivation associated to a.
If $a \in \mathcal{A}$ is such that $X_a = 0$ one says that a is a Casimir (of $\mathcal{A}$).

A subspace $\mathcal{B}$ of $\mathcal{A}$ is called a Poisson ideal if

$$\begin{array}{ccc} \mathcal{B} \cdot \mathcal{A} \subseteq \mathcal{B} & \text{and} & \{\mathcal{B}, \mathcal{A}\} \subseteq \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{B} \text{ is an ideal of } (\mathcal{A}, \cdot) & & \mathcal{B} \text{ is an ideal of } (\mathcal{A}, \{\cdot,\cdot\}). \end{array}$$

$\mathcal{A}$ is said to be (Poisson) simple if its only Poisson ideals are $\mathcal{A}$ and $\{0\}$.

A morphism of Poisson algebras $\varphi : \mathcal{A} \to \mathcal{B}$ is a linear map which is an algebra morhpism and a Li algebra morhpism.

In these lectures we are dealing with polynomial Poisson algebras,

$$(\mathcal{A} = \mathbb{F}[x_1, \ldots, x_n], +, \{\cdot, \cdot\}).$$

A polynomial Poisson bracket $\{\cdot, \cdot\}$ is completely determined by $(\{x_i, x_j\})_{1 \le i,j \le n}$:

$$\text{For } f, g \in \mathcal{A},\ \{f, g\} = \sum_{1 \le i,j \le n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

So if we define a matrix $P = (P_{i,j})$ by $P_{i,j} := \{x_i, x_j\}$ then

$$\{f, g\} = (\nabla f)^T P (\nabla g),$$

where $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)^T$. In particular, g is a Casimir if and only if $P(\nabla g) = 0$.

Warning: Not every choice of $\{x_i, x_j\}$ leads to a Poisson bracket as the Jacobi identity may not be satisfied.

Proposition 2.16. The skew-symmetric biderivation $\{\cdot, \cdot\}$ defined by

$$\{f, g\} = \sum_{i,j=1}^{n} P_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

satisfies the Jacobi identity (hence is a Poisson bracket) if and only if

$$\{\{x_i, x_j\}, x_k\} + \text{cyclic}(i, j, k) = 0,$$

if and only if the Jacobi identity is satisifed for all triplets x_i, x_j, x_k with $i < j < k$.

Proposition 2.17. As in the case of algebras and Lie algebras, the Kernel of a Poisson morphism is a Poisson ideal and every Poisson morphism factors into a surjective and an injective morphisms.

Proposition 2.18. Let $\mathcal{A}$ be a Poisson algebra.

1) The Casimirs of $\mathcal{A}$ form a Poisson subalgebra.
2) The ideal generated by any collection of Casimirs is a Poisson ideal.

Proof.

1) Let c_1 and c_2 be Casimirs of $\mathcal{A}$ then $\{c_1, c_2\} = 0$; but also, for any $a \in \mathcal{A}$,

$$\{a, c_1\, c_2\} = c_1 \underbrace{\{a, c_2\}}_{0} + \underbrace{\{a, c_1\}}_{0} c_2 = 0,$$

thus $c_1\, c_2$ is a Casimir of $\mathcal{A}$.

2) Let $\mathcal{I} = \langle c_j \rangle_{j \in J} \subseteq \mathcal{A}$, where all c_j are Casimirs. For $x \in \mathcal{I}$ we have $x = \sum x_j\, c_j$, for some $x_j \in \mathcal{A}$ and where all but finitely of them are zero. We need to show that for any $a \in \mathcal{A}$, $\{x, a\} \in \mathcal{I}$.

$$\begin{aligned} \{x, a\} &= \left\{\sum x_j\, c_j, a\right\} = \sum \{x_j\, c_j, a\} \\ &= \sum \left(x_j\, \{c_j, a\} + \{x_j, a\}\, c_j\right) \\ &= \sum \{x_j, a\}\, c_j \in \mathcal{I}, \end{aligned}$$

where we have used that $\{c_j, a\} = 0$ since c_j is a Casimir of $\mathcal{A}$, for all $j \in J$.

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Example 2.19.

1) Let $\mathcal{V}$ be an $\mathbb{F}$-vector space with a skew-symmetric bilinear form $\sigma : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$. Let $\mathcal{A} = \mathrm{Sym}(\mathcal{V}) \simeq \mathbb{F}[x_1, \ldots, x_n]$, if we suppose that $\mathcal{V}$ is generated by $x_1, \ldots, x_n$.
σ extends to a unique (skew-symmetric) biderivation of $\mathcal{A}$: $\{x_i, x_j\} := \sigma(x_i, x_j)$.
In order to check the Jacobi identity it is enough to do it for triplets x_i, x_j, x_k. In effect,

$$\{\{x_i, x_j\}, x_k\} + \mathrm{cyclic}(i, j, k) = 0,$$

since $\{x_i, x_j\} \in \mathbb{F}$ and then $\{\{x_i, x_j\}, x_k\} = 0$.

1) Let $\mathfrak{g}$ be a Lie algebra. On $\mathrm{Sym}(\mathfrak{g})$ a Poisson structure is defined as follows:
For $x, y \in \mathfrak{g}$ define $\{x, y\} := [x, y]$ and extend $\{\cdot, \cdot\}$ to a biderivation.

For monomials $\underline{x} = x_1 \cdots x_m$ and $\underline{y} = y_1 \cdots y_n$ this gives

$$\{\underline{x}, \underline{y}\} = \sum_{i,j} \underbrace{[x_i, y_j]}_{\in \mathfrak{g}}\, x_1 \cdots \widehat{x}_i \cdots x_m\, y_1 \cdots \widehat{y}_j \cdots y_n.$$

The Jacobi identity for $\mathfrak{g}$ then implies the corresponding one for $\mathrm{Sym}(\mathfrak{g})$.

Taking $\mathcal{I} = \langle \mathfrak{g} \rangle \subseteq \mathrm{Sym}(\mathfrak{g})$ we get a non-trivial proper ideal of $\mathrm{Sym}(\mathfrak{g})$.

3) (Combination of the previous two examples) If $\mathfrak{g}$ is a Lie algebra and $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ is bilinear, skew-symmetric and satisfies

$$\sigma([x_i, x_j], x_k) + \text{cyclic}(i, j, k) = 0, \quad \text{for every } x_i, x_j, x_k \in \mathfrak{g},$$

then $\mathrm{Sym}(\mathfrak{g})$ is a Poisson algebra with

$$\{x_i, x_j\} := [x_i, x_j] + \sigma(x_i, x_j).$$

4) Let $\varphi, \psi \in \mathcal{A} := \mathbb{F}[x, y, z]$, with $\varphi \neq 0$ and $\psi \notin \mathbb{F}$. On $\mathcal{A}$ a Poisson bracket is defined by

$$\{x, y\} = \varphi \frac{\partial \psi}{\partial z}, \quad \{y, z\} = \varphi \frac{\partial \psi}{\partial x}, \quad \{z, x\} = \varphi \frac{\partial \psi}{\partial y}.$$

The only thing to check is that $\{\{x, y\}, z\} + \text{cyclic}(x, y, z) = 0$. A direct computation shows:

$$\{\{x,y\},z\} = \varphi^2 \left(\frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial z} \right) - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial z} \right) \right) + \varphi \frac{\partial \psi}{\partial z} \left(\frac{\partial \psi}{\partial x} \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \varphi}{\partial x} \right),$$
$$\{\{y,z\},x\} = \varphi^2 \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial x} \right) - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial x} \right) \right) + \varphi \frac{\partial \psi}{\partial x} \left(\frac{\partial \psi}{\partial y} \frac{\partial \varphi}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \varphi}{\partial y} \right),$$
$$\{\{z,x\},y\} = \varphi^2 \left(\frac{\partial \psi}{\partial z} \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial y} \right) \right) + \varphi \frac{\partial \psi}{\partial y} \left(\frac{\partial \psi}{\partial z} \frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \varphi}{\partial z} \right),$$

which sums zero after canceling out by pairs and use the equality of the mixed partial derivatives for the first column terms.

This bracket is called a Nambu-Poisson bracket. For $f, g \in \mathbb{F}[x, y, z]$ one has:

$$\{f, g\} = \varphi \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} & \frac{\partial \psi}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} & \frac{\partial \psi}{\partial y} \\ \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} & \frac{\partial \psi}{\partial z} \end{vmatrix},$$

so ψ is a non-constant Casimir of $\{\cdot, \cdot\}$. Consequently this Nambu-Poisson algebra is not simple.

5) (Generalization) Let $\varphi, \psi_3, \ldots, \psi_n \in \mathcal{A} := \mathbb{F}[x_1, \ldots, x_n]$. For $f, g \in \mathbb{F}[x_1, \ldots, x_n]$,

$$\{f, g\} = \varphi \begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial g}{\partial x_1} & \frac{\partial \psi_3}{\partial x_1} & \cdots & \frac{\partial \psi_n}{\partial x_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n} & \frac{\partial g}{\partial x_n} & \frac{\partial \psi_3}{\partial x_n} & \cdots & \frac{\partial \psi_n}{\partial x_n} \end{vmatrix},$$

defines a Poisson bracket, where $\psi_3, \ldots, \psi_n$ are Casimirs. Hence this Poisson algebra is never simple.

6) Let $A = (a_{ij})$ be a skew-symmetric matrix with entries in $\mathbb{F}$. For $f, g \in \mathcal{A} := \mathbb{F}[x_1, \ldots, x_n]$ we can define

$$\{f, g\} := \sum_{1 \le i,j \le n} a_{ij}\, x_i\, x_j\, \frac{\partial f}{\partial x_i}\, \frac{\partial g}{\partial x_j},$$

i.e., $\{x_i, x_j\} = a_{ij}\, x_i\, x_j$. This is known as a diagonal (quadratic) Poisson bracket. To check the Jacobi identity notice that

$$\{\{x_i, x_j\}, x_k\} = (a_{ij}\, a_{jk} - a_{ji}\, a_{ik})\, x_i\, x_j\, x_k,$$

from where $\{\{x_i, x_j\}, x_k\} + \text{cyclic}(i, j, k) = 0$ easily follows.

7) On $\mathbb{C}[x, y, z]$ take

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y,$$

that is, the Nambu-Poisson bracket with $\varphi = 1$ and $\psi = \frac{1}{2}\,(x^2 + y^2 + z^2)$. As before, ψ is a Casimir. The ideal $\langle \psi \rangle$ is a Poisson ideal, then the nilpotent cone

$$\frac{(\mathbb{C}[x, y, z], \{\cdot, \cdot\})}{\langle x^2 + y^2 + z^2 \rangle}$$

is equipped with a Poisson structure.

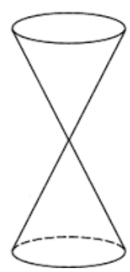

FIGURE 2. Nilpotent cone.

Poisson manifolds

Definition 2.20. A Poisson manifold is a manifold $\mathcal{M}$ for which $\mathcal{C}^\infty(\mathcal{M})$ is eaquipped with a Poisson bracket.

Example 2.21.

- Let $(\mathcal{M}, \omega)$ be a symplectic manifold, *i.e.*, ω is a closed 2-form which is non-degenerate. To each $f \in \mathcal{C}^\infty(\mathcal{M})$ a vector field X_f is assigned:

$$\omega(X_f, \cdot) = df.$$

A Poisson bracket on $\mathcal{C}^\infty(\mathcal{M})$ is defined by

$$\{f, g\} := \omega(X_f, X_g), \quad \text{for } f, g \in \mathcal{C}^\infty(\mathcal{M}).$$

It turns out that a bracket defined as above satisfies the Jacobi identity if and only if ω is a closed 2-form (*i.e.*, $d\,\omega = 0$).

Take for example $\mathcal{M} = \mathbb{R}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$, and the 2-form

$$\omega = \sum_{i=1}^{n} d\,x_i \wedge d\,y_i.$$

$(\mathbb{R}^{2n}, \omega)$ is a symplectic manifold and the Poisson bracket looks like

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right), \quad \text{for } f, g \in \mathcal{C}^\infty(\mathbb{R}^{2n}).$$

Definition 2.22. A Lie-Poisson group is a Lie group G equipped with a Poisson structure such that

$$\begin{array}{ccc} G \times G & \to & G \\ (g, h) & \mapsto & gh \end{array}$$

is a Poisson map.

Application: deformation of commutative algebras

Let us fix some notation:

- $\mathcal{A}$: commutative algebra with unit over $\mathbb{F}$ (a field of characteristic zero).
- ν: formal parameter.
- $\mathcal{A}^\nu = \mathcal{A}[[\nu]]$, and $\mathbb{F}^\nu = \mathbb{F}[[\nu]]$: formal power series.

Definition 2.23. A formal deformation of $\mathcal{A}$ is an $\mathbb{F}^\nu$ (associative) algebra structure $\star$ on $\mathcal{A}^\nu$ such that for $f, g \in \mathcal{A} \subseteq \mathcal{A}^\nu$,

$$f \star g = f \cdot g + \sum_{i>0} \pi_i(f, g)\, \nu^i.$$

So $i = 0$ gives back to the original product.

Let us suppose that $\star$ exists, and in order to simplify the discussion, let us suppose further that π_1 is skew-symmetric.
For $f, g \in \mathcal{A} \subseteq \mathcal{A}^\nu$, $f \star g = f \cdot g + \pi_1(f, g)\, \nu + \mathcal{O}(\nu^2)$.

$$\begin{aligned} [f, g]_\star &:= f \star g - g \star f \\ &= \pi_1(f, g)\, \nu - \pi_1(g, f)\, \nu + \mathcal{O}(\nu^2) \\ &= 2\, \pi_1(f, g)\, \nu + \mathcal{O}(\nu^2). \end{aligned}$$

Since $\star$ is associative then $[\cdot, \cdot]_\star$ is a Lie bracket, so the Jacobi identity holds.

$$\begin{aligned} 0 &= [[f, g]_\star, h]_\star + \text{cyclic}(f, g, h) \\ &= [2\, \pi_1(f, g)\, \nu + \mathcal{O}(\nu^2), h]_\star + \text{cyclic}(f, g, h) \\ &= 4\, \pi_1(\pi_1(f, g), h)\, \nu^2 + \mathcal{O}(\nu^3) + \text{cyclic}(f, g, h). \end{aligned}$$

In particular,

$$\pi_1(\pi_1(f,g),h) + \text{cyclic}(f,g,h) = 0,$$

thus π_1 is a Lie bracket.

What does the associativity of $\star$ imply for π_1?

$$\begin{aligned}
&(f \star g) \star h = f \star (g \star h), \\
&\Rightarrow \big(fg + \pi_1(f,g)\,\nu + \mathcal{O}(\nu^2)\big) \star h = f \star \big(gh + \pi_1(g,h)\,\nu + \mathcal{O}(\nu^2)\big) \\
&\Rightarrow fgh + (\pi_1(f,g)h + \pi_1(fg,h))\,\nu + \mathcal{O}(\nu^2) = fgh + (\pi_1(g,h)\cdot f + \pi_1(f,gh))\,\nu + \mathcal{O}(\nu^2) \\
&\Rightarrow \pi_1(f,g)h - \pi_1(f,gh) + \pi_1(fg,h) - \pi_1(g,h)f = 0. \qquad (2.1)
\end{aligned}$$

After making the cyclic permutation $f \to g \to h$ in (2.1),

$$\pi_1(g,h)f - \pi_1(g,hf) + \pi_1(gh,f) - \pi_1(h,f)g = 0. \qquad (2.2)$$

Substracting (2.2) from (2.1) and reordering terms one obtains:

$$\pi_1(f,gh) = \pi_1(f,g)h + \pi_1(f,h)g,$$

which means that π_1 is a biderivation, hence it is a Poisson bracket.

In general, the skew-symmetric part of π_1, π_1^-, is a Poisson bracket.

In view of the above, several questions arise:

- ⋄ Does associativity imply other conditions on π_i, for $i > 0$?
- ⋄ Given a Poisson algebra $(\mathcal{A}, \cdot, \{\cdot,\cdot\})$, does there exist a deformation of $\mathcal{A}$ such that $f \star g = f \cdot g + \frac{1}{2}\{f,g\}\,\nu + \mathcal{O}(\nu^2)$?
  Answer by Kontsevich in 1992: yes if $\mathcal{A}$ is smooth (*i.e.*, $\Omega^1(\mathcal{A})$ is projective over $\mathcal{A}$).

Similar questions:

- ⋄ Can one deform $\{\cdot,\cdot\}$?
- ⋄ $\mathcal{A}^\nu$ viewed as a commutative algebra over $\mathbb{F}^\nu$; is $\{\cdot,\cdot\}_\star = \sum_{k\geq 0} \frac{1}{k!}\{\cdot,\cdot\}\,\nu^k$ an $\mathbb{F}^\nu$-Poisson bracket?

Poisson cohomology

Let $(\mathcal{A}, \cdot, \{\cdot,\cdot\})$ be a Poisson algebra. A p-derivation on $\mathcal{A}$ is a p-linear map $P : \mathcal{A}^p \to \mathcal{A}$ which is a derivation in each argument. Let us define

$$\mathfrak{X}^p(\mathcal{A}) := \{P : \mathcal{A}^p \to \mathcal{A} \mid P \text{ is a skew-symmetric } p\text{-derivation}\}.$$

On $\mathfrak{X}^\bullet(\mathcal{A}) := \bigoplus_{p=0}^{\infty} \mathfrak{X}^p(\mathcal{A})$ there are two operations:

▷ Wedge product: for $P \in \mathfrak{X}^p(\mathcal{A})$ and $Q \in \mathfrak{X}^q(\mathcal{A})$, $P \wedge Q$ is defined by

$$P \wedge Q(f_1, \ldots, f_{p+q}) := \sum_{\sigma \in S_{p,q}} \operatorname{sign}(\sigma)\, P(f_{\sigma(1)}, \ldots, f_{\sigma(p)})\, Q(f_{\sigma(p+1)}, \ldots, f_{\sigma(p+q)}),$$

where $S_{p,q}$ denotes the set of shuffle permutations of $\{1, 2, \ldots, p+q\}$, that is, permutations which satisfy

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p), \quad \text{and} \quad \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q).$$

It can be checked that $P \wedge Q \in \mathfrak{X}^{p+q}(\mathcal{A})$.

▷ Schouten bracket: for $P \in \mathfrak{X}^p(\mathcal{A})$ and $Q \in \mathfrak{X}^q(\mathcal{A})$, $[P, Q]_S$ is defined by

$$\begin{aligned}[P, Q]_S(f_1, \ldots, f_{p+q-1}) := &\sum_{\sigma \in S_{q,p-1}} \operatorname{sign}(\sigma) P(Q(f_{\sigma(1)}, \ldots, f_{\sigma(q)}), f_{\sigma(q+1)}, \ldots, f_{\sigma(p+q-1)}) \\ &+ \sum_{\sigma \in S_{p,q-1}} \operatorname{sign}(\sigma) Q(P(f_{\sigma(1)}, \ldots, f_{\sigma(p)}), f_{\sigma(p+1)}, \ldots, f_{\sigma(p+q-1)}).\end{aligned}$$

It can also be checked that $[P, Q]_S \in \mathfrak{X}^{p+q-1}(\mathcal{A})$.

With the above operations it turns out that $(\mathfrak{X}^\bullet(\mathcal{A}), [\cdot, \cdot]_S)$ is a graded Lie algebra.

Remark 2.24. For $P \in \mathfrak{X}^2(\mathcal{A})$, $[P, P]_S = 0$ if and only if P satisfies the Jacobi identity.

The graded Jacobi identity:

$$(-1)^{(p-1)(r-1)}[[P, Q]_S, R]_S + \operatorname{cyclic}(P, Q, R) = 0,$$

for every $P \in \mathfrak{X}^p(\mathcal{A})$, $Q \in \mathfrak{X}^q(\mathcal{A})$ and $R \in \mathfrak{X}^r(\mathcal{A})$.

If $\Pi \in \mathfrak{X}^2(\mathcal{A})$ is a Poisson structure on $\mathcal{A}$, *i.e.*, $[\Pi, \Pi]_S = 0$, then from the graded Jacobi identity we get

$$[\Pi, [\Pi, P]_S]_S = 0,$$

for every $P \in \mathfrak{X}^p(\mathcal{A})$. Thus if we define $\delta_\Pi : \mathfrak{X}^\bullet(\mathcal{A}) \to \mathfrak{X}^{\bullet+1}(\mathcal{A})$ by $\delta_\Pi(P) := [\Pi, P]_S$, the last equation can be written as $\delta_\Pi \circ \delta_\Pi = 0$. Consequently we obtain a cochain complex whose homology is called the Poisson cohomology of $\mathcal{A}$.