DECAY OF SOLUTIONS OF DISPERSIVE EQUATIONS AND POISSON BRACKETS IN ALGEBRAIC GEOMETRY

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Presentation

Part I

DECAY OF SOLUTIONS OF DISPERSIVE EQUATIONS

Presentation

Our starting point will be the initial value problem (IVP) associated to the Korteweg-de Vries (KdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad u = u(x,t), \quad x,t \in \mathbb{R}, \\ u(0) = u_0. \end{cases} \tag{1}$$

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We intend to study a decay property of exponential type of its solutions $u(x,t)$ in the positive semiaxis x.

Results on the local and global well-posedness for the IVP (1) in the context of Sobolev spaces $H^s(\mathbb{R})$ have been obtained and successively improved in a series of papers of which we cite among others

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Saut and Temam [**ST**], Bona and Smith [**BS**], Bona and Scott [**BSc**], Kato [**K**], Kenig, Ponce and Vega [**KPV1**], [**KPV2**], Bourgain [**B**], Colliander, Keel, Staffilani, Takaoka and Tao [**CKSTT**], Christ, Colliander and Tao [**CCT**], Guo [**G**], and Kishimoto [**Ki**].

In **[EKPV]**, Escauriaza, Kenig, Ponce and Vega showed that there exists a constant $a_0 > 0$ such that if $a > a_0$ and if a solution u of the IVP (1) satisfies

$$e^{a\, x_+^{3/2}} u(0) \in L^2(\mathbb{R}) \quad \text{and} \quad e^{a\, x_+^{3/2}} u(1) \in L^2(\mathbb{R}),$$

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then $u \equiv 0$.

The exponent of order $x^{3/2}$ is related to the decay of the fundamental solution of the IVP (1).

Presentation

The question arises about if, for an initial datum u_0 with $e^{a_0 x_+^{3/2}} u_0 \in L^2(\mathbb{R})$, the solution of the IVP (1) keeps some decay with exponent of order $x_+^{3/2}$ as time evolves.

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An affirmative answer to this question was given in **[ILP]**, where it was proved, that if $e^{a_0 x_+^{3/2}} u_0 \in L^2(\mathbb{R})$, then the solution $u(t)$ on $[0, T]$ is such that

$$\left\| e^{a(t)\, x_+^{3/2}} u(t) \right\|_{L^2(\mathbb{R})} \leq C, \tag{2}$$

where $C = C\left(a_0, T, \|u_0\|_{L^2(\mathbb{R})}, \|e^x u_0\|_{L^2(\mathbb{R})}\right)$, and

$$a(t) = \frac{a_0}{\sqrt{1 + 27\, a_0^2 t}}, \qquad t \in [0, T].$$

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$$a(t) = \frac{a_0}{\sqrt{1 + 27\, a_0^2 t}}, \qquad t \in [0,T].$$

Our purpose is to obtain an optimal function $a(t)$, with $a(0) = a_0$ for which (2) holds if $e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R})$.

We analize the behavior of the fundamental solution of the linear problem associated to IVP (1),

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which is

$$S_t(x) = \frac{1}{\sqrt[3]{3t}}\, A\left(\frac{x}{\sqrt[3]{3t}}\right),$$

where A is the Airy function.

It is known that, for $x > 0$,

$$S_t(x) \sim x^{-1/4}\, t^{-1/4}\, e^{-\frac{2}{3\sqrt{3}}\,\frac{x^{3/2}}{\sqrt{t}}}\,.$$

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The previous exponent is $a_0 x^{3/2}$ at the instant $t_0 = \frac{27}{4a_0^2}$. If we take t_0 as the initial time and measure time t from that instant on, the fundamental solution at t will be

$$u(t)(x) = S_{t_0+t}(x) \sim x^{-1/4}\,(t_0+t)^{-1/4}\, e^{-\frac{a_0}{\sqrt{1+\frac{27}{4}a_0^2 t}}\, x^{3/2}}, \qquad x > 0,\ t > -t_0.$$

In this thesis we will prove that the function

$$a(t) = \frac{a_0}{\sqrt{1 + \frac{27}{4}\, a_0^2 t}}$$

produces the optimal decay of exponential order $3/2$ to the right of the x-axis, as t evolves, when the initial datum satisfies $e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R})$.

More precisely:

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Theorem I

For $u_0 \in L^2(\mathbb{R})$ *and* $T > 0$, *let* $u \in C\left([0,T]; L^2(\mathbb{R})\right)$ *be the solution of the IVP* (1) *with* $u(0) = u_0$. *Let us suppose that for* $a_0 > 0$,

$$e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R}).$$

Then

$$\left\| e^{a(t)\, x_+^{3/2}} u(t) \right\|_{L^2(\mathbb{R})} \leq C \left\| e^{a_0\, x_+^{3/2}} u_0 \right\|_{L^2(\mathbb{R})}, \qquad \text{for every } t \in [0,T],$$

where

$$a(t) = \frac{a_0}{\sqrt{1 + \frac{27}{4} a_0^2 t}}, \qquad t \in [0,T],$$

and $\quad C = C\left(a_0, T, \|u_0\|_{L^2(\mathbb{R})}, \|e^x u_0\|_{L^2(\mathbb{R})}\right).$

Our second result establishes that the function $a(t)$ obtained in Theorem I is optimal.

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Theorem II

For $T > 0$, $a_0 > 0$ *and* $0 < \epsilon < \frac{1}{3}\, a_0$, *there exist* $u_0 \in \mathcal{S}(\mathbb{R})$ *with* $e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R})$ *and* $C > 0$ *such that the solution* u *on* $[0,T]$ *of the IVP* (1) *with initial datum* u_0 *satisfies*

$$C\, e^{-g(t)(a_0+\epsilon)\, x^{3/2}} \leq u(t)(x), \qquad \textit{for every } t \in [0,T] \textit{ and every } x > 0.$$

In particular, $e^{g(t)(a_0+\epsilon)\, x_+^{3/2}}\, u(t) \notin L^2(\mathbb{R})$, *for every* $t \in [0,T]$.

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We consider a function $\rho \in C_0^\infty(\mathbb{R})$ with $\rho \geq 0,\ supp(\rho) \subset [-1,1]$ and such that

$$\int_{\mathbb{R}} \rho \, dx = 1.$$

For $\epsilon \in (0,1)$, we define

$$\rho_\epsilon := \frac{1}{\epsilon}\,\rho\left(\frac{\cdot}{\epsilon}\right) \quad \text{and}$$

$$u_0^\epsilon(x) := \rho_\epsilon * u_0(\cdot + \epsilon)(x) = \int_{\mathbb{R}} \rho_\epsilon(y)\, u_0(x+\epsilon-y)\, dy.$$

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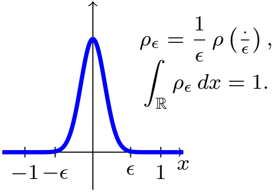

Presentation

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We are going to make an a priori estimate of $u \equiv u_m$. For this, let us take a truncation function $\omega \in C^\infty(\mathbb{R})$, like it is shown in the next figure:

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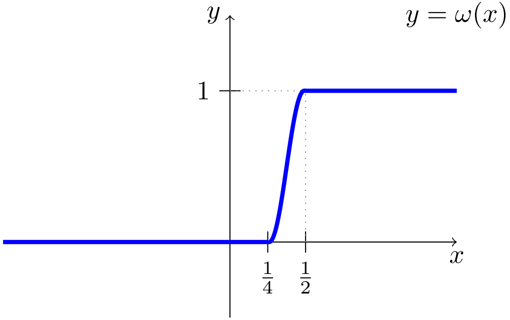

For each positive integer n, we consider a function ψ_n defined in the following fashion:

$$\psi(x,t) \equiv \psi_n(x,t) := \begin{cases} \omega(x)\, a(t)\, x^{3/2}, & \text{if} \quad x \leq n, \\ \log\left(P_n(x,t)\right), & \text{if} \quad x > n, \end{cases}$$

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where, for fixed $t \in [0,T]$, $P_n(x,t)$ is the second degree polynomial in x which coincides with $e^{\omega(x)\, a(t)\, x^{3/2}} = e^{a(t)\, x^{3/2}}$ at $x = n$ together with its two first derivatives.

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For a fixed positive integer m and for $n \in \mathbb{N}$, let us define

$$f \equiv f_{m,n} = u_m\, e^{\psi_n} = u\, e^{\psi}.$$

Then, $u = e^{-\psi} f$. We replace u in the KdV equation and multiply by e^{ψ}. Next we multiply by f and integrate by parts with respect to the variable x, on $\mathbb{R}$, to obtain:

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$$\frac{1}{2}\frac{d}{dt}\int f^2 + 3\int \psi_x(\partial_x f)^2 - \int(\psi_t + \psi_x^3 + \psi_{xxx})f^2 - \frac{2}{3}\int e^{-\psi}\psi_x f^3 = 0.$$

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Hence,

$$\frac{1}{2}\frac{d}{dt}\int f^2 \leq \int(\psi_t + \psi_x^3 + \psi_{xxx})f^2 + \frac{2}{3}\int e^{-\psi}\psi_x f^3. \qquad \clubsuit$$

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Our objective is to apply Gronwall's lemma to estimate $\int f^2$.

We start by studying the terms on the right hand side of ♣ for $1 \leq x \leq n$, where we know that $\psi = a\, x^{3/2}$. Then, the first integrand is

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whose solution is given by

$$a(t) = \frac{a_0}{\sqrt{1 + \frac{27}{4} a_0^2 t}}.$$

In the interval $[1, n]$ we find that the integrals on the right hand side of ♣ are bounded by

$$a_0 \, \|x_+^{1/2} u(t)\|_{L^\infty([0,\infty))} \int_{\mathbb{R}} f^2.$$

Next, we consider the contribution of the interval (n, ∞) to the integrals of the right hand side of expression $\clubsuit$.

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From the definition of the polynomial $P \equiv P_n$, we have that

$$P(x,t) = \left[1 + \frac{3}{2}\, a\, n^{1/2}(x-n) + \left(\frac{3}{8}\, a\, n^{-1/2} + \frac{9}{8}\, a^2 n\right)(x-n)^2\right] e^{a\, n^{3/2}},$$

$$P_t(x,t) = a' \left[n^{3/2} + \left(\frac{3}{2}\, n^{1/2} + \frac{3}{2}\, a\, n^2\right)(x-n) + \left(\frac{3}{8}\, n^{-1/2} + \frac{21}{8}\, a\, n + \frac{9}{8}\, a^2 n^{5/2}\right)(x-n)^2\right] e^{a\, n^{3/2}},$$

$$P_x(x,t) = \left[\frac{3}{2}\, a\, n^{1/2} + \left(\frac{3}{4}\, a\, n^{-1/2} + \frac{9}{4}\, a^2 n\right)(x-n)\right] e^{a\, n^{3/2}},$$

$$P_{xx}(x,t) = \left(\frac{3}{4}\, a\, n^{-1/2} + \frac{9}{4}\, a^2 n\right) e^{a\, n^{3/2}}.$$

Seeing the previous polynomials as perturbations of certain polynomials in the variable $r := a\, n^{1/2}(x-n)$, and using a continuity argument we find that for $x \in (n,\infty)$ and $n > N$ (where N is certain large positive integer),

$$\psi_t + \psi_x^3 + \psi_{xxx} = \frac{1}{P^3}\left[P^2 P_t + 3\, P_x^3 - 3\, P\, P_x P_{xx}\right] < 0.$$

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Furthermore,

$$\left|\frac{2}{3}\int_n^\infty e^{-\psi}\psi_x\, f^3\right| \le (1+a_0)\left\|x_+^{1/2}u(t)\right\|_{L^\infty([0,\infty))}\int_{\mathbb{R}} f^2.$$

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Hence, the integrals on the right hand side of ♣ performed on the interval (n,∞) are bounded by

$$(1+a_0)\left\|x_+^{1/2}u(t)\right\|_{L^\infty([0,\infty))}\int_{\mathbb{R}} f^2.$$

Putting together all the results obtained on the right hand side of ♣ we conclude that

$$\frac{1}{2}\frac{d}{dt}\int f^2 \leq C\,(1+a_0^3)\left(1+\|(1+x_+^{1/2})u(t)\|_{L^\infty([0,\infty))}\right)\int f^2.$$

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Since $1+x_+^{1/2} \le 2e^{x_+}$, using a Sobolev embedding theorem and returning to the notation $u = u_m$, we have that

$$\frac{1}{2}\frac{d}{dt}\int e^{2\,\psi_n}u_m(t)^2\,dx \le \beta_m(t)\int e^{2\,\psi_n}u_m(t)^2\,dx,$$

where $\beta_m(t) = C\,(1+a_0^3)\left(1+\|e^x u_m(t)\|_{L^2(\mathbb{R})}+\|e^x\partial_x u_m(t)\|_{L^2(\mathbb{R})}\right)$, and C is a universal constant.

It can be seen that

$$\int_0^T \beta_m(s)\,ds \leq C\,(1+a_0^3)(1+T)\,e^{KT}\,\|e^x u_0\|_{L^2(\mathbb{R})}\,,$$

where K is some constant which only depends on $\|u_0\|_{L^2(\mathbb{R})}$.

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where K is some constant which only depends on $\|u_0\|_{L^2(\mathbb{R})}$.

Applying Gronwall's lemma, we conclude that

$$\begin{aligned}\int e^{2\,\psi_n(x,t)} u_m(t)^2\,dx &\le \exp\left(\int_0^T \beta_m(s)\,ds\right)\int e^{2\,\psi_n(x,0)} u_m(0)^2\,dx\\ &\le C\int e^{2\,\psi_n(x,0)} u_m(0)^2\,dx,\end{aligned}$$

where $C = C\left(a_0, T, \|u_0\|_{L^2(\mathbb{R})}\,, \|e^x u_0\|_{L^2(\mathbb{R})}\right)$.

We apply Fatou's lemma, letting $n \to \infty$, to get that

$$\int_{\mathbb{R}} \left(e^{a(t)\, x_+^{3/2}} u_m(t)\right)^2 dx \leq C \int_{\mathbb{R}} \left(e^{a_0\, x_+^{3/2}} u_0\right)^2 dx,$$

for every $t \in [0, T]$, and for all $m \in \mathbb{N}$.

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for every $t \in [0,T]$, and for all $m \in \mathbb{N}$.

There is a subsequence $u_{m_j}(t)$ such that $u_{m_j}(t)(x) \to u(t)(x)$, for almost every $x \in \mathbb{R}$, when $j \to \infty$. Thus, applying Fatou's lemma once again, for this subsequence, we obtain that

$$\int_{\mathbb{R}} \left(e^{a(t)\, x_+^{3/2}} u(t) \right)^2 dx \leq C \int_{\mathbb{R}} \left(e^{a_0\, x_+^{3/2}} u_0 \right)^2 dx, \quad \text{for all } t \in [0,T].$$

For the linear problem associated to the IVP (1), we can obtain a result similar to that in Theorem I.

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If u is a solution of the linear problem associated to (1) and $e^{a_0\, x_+^{3/2}} u(0) \in L^2(\mathbb{R})$, then

$$\left\| e^{a(t)\, x_+^{3/2}} u(t) \right\|_{L^2(\mathbb{R})} \leq C \left\| e^{a_0\, x_+^{3/2}} u(0) \right\|_{L^2(\mathbb{R})},$$

where $C = C(a_0, T)$.

Presentation

In the proof of Theorem II we will construct a function of the Schwartz class which satisfies the conclusions of the theorem. For this function it will be important to study the exponential decay of order 3/2 of its first and second derivatives.

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Proposition

For $T > 0$ and $u_0 \in \mathcal{S}(\mathbb{R})$ let u be the solution on $[0,T]$ of the IVP (1) with initial datum u_0. If $a_0 \geq 0$ and $e^{a_0\, x_+^{3/2}} \partial_x^j u_0 \in L^2(\mathbb{R})$, then there exist constants C and M, such that

$$\left\| e^{a(t)\, x_+^{3/2}} \partial_x^j u(t) \right\|_{L^2(\mathbb{R})} \leq e^{MT} \left\| e^{a_0\, x_+^{3/2}} \partial_x^j u_0 \right\|_{L^2(\mathbb{R})}, \quad \text{for every } t \in [0,T],$$

where $j = 1,2$, and

$$M = C(1 + a_0^3) \sup_{t\in[0,T]} \left[1 + \|(1 + x_+^{1/2}) u(t)\|_{L^\infty([0,\infty))} + \|\partial_x u(t)\|_{L^\infty(\mathbb{R})} \right],$$

and C is an absolute constant.

Presentation

Let us take a function $\varphi \in C_c^\infty(\mathbb{R})$ such that $\varphi \geq 0$, $supp(\varphi) \subset (-1,1)$ and $\int_{\mathbb{R}} \varphi = 1$, and, for $\delta \in (0,1/2)$ let $\varphi_\delta = \frac{1}{\delta}\varphi\left(\frac{\cdot}{\delta}\right)$.

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For $\alpha > 0$ small, which will be properly chosen later, we consider problem (1) with initial datum

$$u_{0,\alpha} \equiv u_0 = S(t_0)(\alpha\varphi_\delta),$$

where

$$t_0 = \frac{4}{27(a_0 + \epsilon/3)^2}.$$

From the theory of global well-posedness on spaces $H^s(\mathbb{R})$, for $T > 0$, the IVP (1) has a unique solution $u_\alpha \equiv u \in C\left([0,T];\mathcal{S}(\mathbb{R})\right)$, which, given its regularity, satisfies the Duhamel's formula pointwise; that is, From the theory of global well-posedness on spaces $H^s(\mathbb{R})$, for $T > 0$, the IVP (1) has a unique solution $u_\alpha \equiv u \in C\left([0,T]; \mathcal{S}(\mathbb{R})\right)$, which, given its regularity, satisfies the Duhamel's formula pointwise; that is,

$$\begin{aligned} u(t) &= S(t)\, u_0 - \int_0^t S(t-\tau)\left(u(\tau)\partial_x u(\tau)\right)\, d\tau \\ &\equiv S(t)u_0 - F(t), \quad \text{for every } t \in [0,T], \end{aligned}$$

Using the properties of the convolution, the asymptotic behavior of the Airy function, and the previous Remarks, we prove that for $\delta > 0$ small enough, $x > 1$ and $t \in [0, T]$, Using the properties of the convolution, the asymptotic behavior of the Airy function, and the previous Remarks, we prove that for $\delta > 0$ small enough, $x > 1$ and $t \in [0, T]$,

$$\overline{C}\,\alpha\, e^{-g(t)\left(a_0^+\right)x^{3/2}} \leq \left[S(t)u_0\right](x) \leq C\,\alpha\, e^{-g(t)(a_0+\epsilon/4)x^{3/2}},$$

where C and $\overline{C}$ are independent of α and of $t \in [0, T]$.

Our next step is to show that the integral term $F(t)$ in the Duhamel's formula decays as $\alpha^2 e^{-\beta\, x^{3/2}}$, for some $\beta > g(t)\left(a_0^+\right)$, for every $t \in [0, T]$.

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Let us fix a_1 and a_2 such that $a_0^- < a_2 < a_1 < a_0$.

Let us recall that the integral term $F(t)$ in the Duhamel's formula is given by

$$F(t) = \int_0^t S(t-\tau)\,(u(\tau)\partial_x u(\tau))\ d\tau, \qquad t \in [0,T].$$

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$$F(t) = \int_0^t S(t-\tau)\left(u(\tau)\partial_x u(\tau)\right)\, d\tau, \qquad t \in [0,T].$$

In order to estimate $F(t)$, we first analyze $\partial_x F(t)$:

$$\partial_x F(t) = \int_0^t S(t-\tau) f(\tau)\, d\tau,$$

where $f(\tau) \equiv \partial_x \left(u(\tau)\partial_x u(\tau)\right) = \left(\partial_x u(\tau)\right)^2 + u(\tau)\partial_x^2 u(\tau)$.

In virtue of the Fundamental Theorem of Calculus, Fubini's Theorem, Cauchy-Schwarz inequality, and some estimates, we obtain that for $x \geq 0$

$$|F(t)(x)| \leq C\,\alpha^2\,T\,e^{-g(t)(2a_0^-)\,x^{3/2}}.$$

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$$|F(t)(x)| \leq C\,\alpha^2\,T\,e^{-g(t)(2a_0^-)\,x^{3/2}}.$$

Let us notice that if $\epsilon < \frac{1}{3}\,a_0$, then $2a_0^- > a_0^+$. Therefore, from the last estimate,

$$F(t)(x) \leq C\,\alpha^2 e^{-g(t)(a_0^+)\,x_+^{3/2}},$$

with C independent of $x > 1$, $t \in [0,T]$, and of α.

it follows that for $x > 0$,

$$u(t)(x) \geq \overline{C}\,\alpha e^{-g(t)(a_0^+)\,x^{3/2}} - C\,\alpha^2 e^{-g(t)(a_0^+)\,x^{3/2}},$$

where C and $\overline{C}$ do not depend upon $x > 0$, $t \in [0, T]$, and $\alpha > 0$.

it follows that for $x > 0$,

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where C and $\overline{C}$ do not depend upon $x > 0$, $t \in [0, T]$, and $\alpha > 0$.

Thus, by taking $\alpha = \overline{C}/2C$, we obtain that, for $x > 0$

$$u(t)(x) \geq \frac{\overline{C}^2}{4C} e^{-g(t)(a_0^+)x^{3/2}}.$$

Part II

POISSON BRACKETS IN ALGEBRAIC GEOMETRY

Presentation

Let $\mathcal{V}$ be a vector space. A 2-covector ω on $\mathcal{V}$ is said to be *non-degenerate* if for every nonzero vector $v \in \mathcal{V}$, there exists $w \in \mathcal{V}$ such that $\omega(v, w) \neq 0$.

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Example

Let $\mathcal{V}$ be a real vector space of dimension $2n$, and let us fix a basis $\{A_1, B_1, \ldots, A_n, B_n\}$ for $\mathcal{V}$. Let $\{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n\}$ be the corresponding dual basis, for $\mathcal{V}^*$, and $\omega \in \bigwedge^2(\mathcal{V}^*)$ be the 2-covector defined by

$$\omega = \sum_{j=1}^{n} \alpha_j \wedge \beta_j.$$

$(\mathcal{V}, \omega)$ is a symplectic vector space.

Proposition (Canonical form for a symplectic tensor)

Let ω be a symplectic tensor on a vector space $\mathcal{V}$ over $\mathbb{R}$, of dimension m. Then $\mathcal{V}$ has even dimension $m = 2n$, and there exists a basis $\{A_1, B_1, \ldots, A_n, B_n\}$ for $\mathcal{V}$ such that

$$\omega = \sum_{j=1}^{n} \alpha_j \wedge \beta_j,$$

where $\{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n\}$ is the corresponding dual basis.

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$$\omega = \sum_{j=1}^{n} \alpha_j \wedge \beta_j,$$

where $\{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n\}$ is the corresponding dual basis.

Proposition

Let $\mathcal{V}$ be a vector space of dimension $2n$, and $\omega \in \bigwedge^2(\mathcal{V}^*)$. Then ω is a symplectic tensor if and only if $\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$.

Presentation

A *symplectic manifold* is a smooth manifold M with a non-degenerate closed 2-form.

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Example (Local model)

Let $M = \mathbb{R}^{2n}$ with the standard coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The form

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$$

is symplectic.

Let Q be any n-dimensional manifold, and $M = T^*Q$ its cotangent bundle. Let us say that $(T^*U, q_1, \ldots, q_n, p_1, \ldots, p_n)$ is a coordinate chart for M.

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We define a 2-form on T^*U by

$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j .$$

In fact, if we consider the 1-form on T^*U given by

$$\tau = \sum_{j=1}^{n} p_j \, dq_j ,$$

then $\omega = -d\tau$.

Theorem (Darboux's theorem)

Let (M, ω) be a symplectic manifold of dimension $2n$. Then, each point $p \in M$ has a coordinate neighborhood U, with local coordinates $x_1, \dots, x_n, y_1, \dots, y_n$, such that

$$\omega|_U = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

Presentation

Let X be a vector field on M. From Cartan's formula,

$$\mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X(\underbrace{d\omega}_{0}) = d\,\iota_X \omega.$$

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- X is a *symplectic* vector field if $\iota_X \omega$ is a closed 1-form.
- X is a *Hamiltonian* vector field if $\iota_X \omega$ is an exact 1-form.

Given a smooth function $f \in \mathcal{C}^\infty(M;\mathbb{R})$, the *Hamiltonian vector field associated to* f is the unique vector field X_f on M which satisfies

$$\omega(X_f, \cdot) = df.$$

We have an operation, called the *Poisson bracket* on $\mathcal{C}^\infty(M)$,

$$\{\,,\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M),$$

which is given by

$$\{f, g\} := \omega(X_f, X_g),$$

for every $f, g \in \mathcal{C}^\infty(M)$.

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In Darboux's coordinates:

$$X_H=\sum_{j=1}^{n}\left(\frac{\partial H}{\partial y_j}\frac{\partial}{\partial x_j}-\frac{\partial H}{\partial x_j}\frac{\partial}{\partial y_j}\right).$$

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$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial x_j} \right).$$

In a Hamiltonian system (M, ω, H) we have the notions of

- A *conserved quantity.*
- An *infinitesimal symmetry.*

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- A *conserved quantity*.
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Theorem (Noether's theorem)

Let (M, ω, H) be a Hamiltonian system. If f is any conserved quantity, then its Hamiltonian vector field X_f is an infinitesimal symmetry. Conversely, if $\mathcal{H}^1_{dR}(M) = 0$, then each infinitesimal symmetry is the Hamiltonian vector field of a conserved quantity.

Presentation

Definition

A *Poisson algebra* is an $\mathbb{F}$-vector space A equipped with two binary operations: $\cdot\,, \{\,,\} : A \times A \to A$, such that

- $(A, \cdot)$ is a commutative associative algebra over $\mathbb{F}$, with 1.
- $(A, \{\,,\})$ is a Lie algebra over $\mathbb{F}$.
- Both structures are compatible in the sense that

$$\{f \cdot g, h\} = f \cdot \{g, h\} + g \cdot \{f, h\}, \quad \text{for every } f, g, h \in A. \tag{3}$$

In this case the Lie bracket $\{\,,\}$ is called a *Poisson bracket*.

Definition

Let $(A, \cdot, \{\,,\})$ be a Poisson algebra and let $B \subseteq A$ be a vector subspace. Then

- B is a *Poisson subalgebra* of A if it is a subalgebra and a Lie subalgebra of A. That is,

$$B \cdot B \subseteq B \quad \text{and} \quad \{B, B\} \subseteq B.$$

- B is a *Poisson ideal* of A if it is an ideal and a Lie ideal of A. That is,

$$B \cdot A \subseteq B \quad \text{and} \quad \{B, A\} \subseteq B.$$

Fix $H \in A$. The derivation $X_H := \{\cdot, H\}$ of A is called a *Hamiltonian derivation*. We define

$$Ham(A) := \{X_H \mid H \in A\}.$$

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$$Ham(A) := \{X_H \mid H \in A\}.$$

An element $H \in A$ is a Casimir if $X_H(f) = \{f, H\} = 0$, for every $f \in A$. The set of this elements is denoted by

$$Cas(A) := \{H \in A \mid \{f, H\} = 0, \text{ for every } f \in A\}.$$

Proposition

Let $(A, \cdot, \{\,,\})$ *be a Poisson algebra.*

(1) $Cas(A)$ *is a subalgebra of* $(A, \cdot)$*, which contains the image of* $\mathbb{F}$ *in* A*, under the natural inclusion* $a \mapsto a \cdot 1$*.*

(2) *If* A *has no zero divisors, then* $Cas(A)$ *is integrally closed in* A*.*

(3) $Ham(A)$ *is not an* A*-module (in general). Instead,*

$$X_{f \cdot g} = f\,X_g + g\,X_f, \quad \textit{for every } f, g \in A.$$

(4) $Ham(A)$ *is a* $Cas(A)$*-module.*

(5) *The map* $A \to \mathfrak{X}^1(A)$*, defined by* $H \mapsto -X_H$ *is a morphism of Lie algebras. As a consequence,* $Ham(A)$ *is a Lie subalgebra of of* $\mathfrak{X}^1(A)$*.*

(6) *The Lie algebra sequence*

$$0 \longrightarrow Cas(A) \longrightarrow A \xrightarrow{-X} Ham(A) \longrightarrow 0$$

is a short exact sequence.

Presentation

Definition

Let M be an affine variety and suppose that $\mathcal{F}(M)$ is equipped with a Lie bracket $\{\,,\}:\mathcal{F}(M)\times\mathcal{F}(M)\to\mathcal{F}(M)$, which makes $(\mathcal{F}(M),\cdot,\{\,,\})$ into a Poisson algebra.

Definition

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Definition

For a Poisson variety $(M, \{\,,\})$ and $p \in M$, the rank of the Poisson matrix of $\{\,,\}$ at p, is called the *rank* of $\{\,,\}$ at p, denoted by $Rk_p\{\,,\}$. The *rank* of $\{\,,\}$, denoted by $Rk\{\,,\}$ is the maximum $\max_{p \in M} Rk_p\{\,,\}$.

Proposition

Let $(M, \{\,,\})$ be an affine Poisson variety.

(i) *For every $p \in M$, $Rk_p\{\,,\}$ is an even number.*

(ii) *For each $s \in \mathbb{N}$, let us define*

$$M_{(s)} := \{p \in M \mid Rk_p\{\,,\} \geq 2s\} \subseteq M.$$

Then $M_{(s)}$ is open. In particular, the set $\mathcal{U} := \{p \in M \mid Rk_p\{\,,\} = Rk\{\,,\}\}$ is open and dense in M.

(ii) *$Rk\{\,,\}$ is at most equal to the dimension of M.*

Definition (Poisson manifold)

Let Π be a bivector field on a manifold M. Π is a Poisson structure on M if for every open subset $U \subseteq M$, the restriction of Π to U makes $\mathcal{F}(U)$ into a Poisson algebra..

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In bracket notation, $\{f, g\} = \Pi(f, g)$, where $f, g \in \mathcal{F}(U)$. Π can also be written as

$$\Pi = \sum_{1 \le j < k \le d} \{x_j, x_k\} \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}.$$

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$$\Pi = \sum_{1 \le j < k \le d} \{x_j, x_k\} \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}.$$

Given a bivector field Π on M, a necessary and sufficient condition for Π to define a Poisson structure is that $[\Pi, \Pi]_S = 0 \in \mathfrak{X}^3(M)$, where $[\,,]_S$ denotes the *Schouten-Nijenhuis* bracket.

Theorem (Weinstein's splitting theorem)

Let (M, Π) be a Poisson manifold. Let $x \in M$ be an arbitrary point and denote the rank of Π at x by r. There exists a coordinate neighborhood U of x with coordinates $q_1, \dots, q_r, p_1, \dots, p_r, z_1, \dots, z_s$, centered at x, such that, on U,

$$\Pi = \sum_{j=1}^{r} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial p_j} + \sum_{1 \le k,l \le s} \varphi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l},$$

where the functions φ_{kl} are (smooth or holomorphic) functions which depend on $z = (z_1, \dots, z_s)$ only, and which vanish when $z = 0$.

Example

A prime example of a Poisson manifold is that of a symplectic manifold (M, ω), that is, ω is a non-degenerate closed 2-form.

$$\{f, g\} := \omega(X_f, X_g).$$

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Example

Consider the Lie algebra $\mathfrak{g}$ of a Lie group G. The Poisson bracket on $\mathfrak{g}^*$ is given by

$$\{f, h\}(\varphi) = \langle \varphi, [df|_\varphi, dh|_\varphi] \rangle,$$

for $f, h \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and $\varphi \in \mathfrak{g}^*$.

Presentation

A k-dimensional distribution E on M is a datum of a k-dimensional subspace E_p of T_pM, for every $p \in M$.

A k-dimensional distribution E on M is a datum of a k-dimensional subspace E_p of T_pM, for every $p \in M$.

The *Frobenius theorem* states that if E is a (smooth or holomorphic) k-dimensional distribution, then the following conditions are equivalent:

- E is involutive.
- E is completely integrable.
- E arises from a k-dimensional foliation on M.

Presentation

Definition

Let $(M, \{\,,\})$ be an affine Poisson variety and let $\mathcal{A}$ be a subalgebra of $\mathcal{F}(M)$.

- $\mathcal{A}$ is called *involutive* if $\{\mathcal{A}, \mathcal{A}\} = 0$.
- We say that $\mathcal{A}$ is *complete* if for any $f \in \mathcal{F}(M)$ one has $\{f, \mathcal{A}\} = 0$ if and only if $f \in \mathcal{A}$.

The triplet $(M, \{\,,\}, \mathcal{A})$, where $\mathcal{A}$ has the above two properties is called a *complete involutive Hamiltonian system.*

Proposition

Let $(M, \{\,,\}, \mathcal{A})$ *be a complete involutive Hamiltonian system. Then*

$$dim(\mathcal{A}) \leq dim(M) - \frac{1}{2} Rk\,\{\,,\}.$$

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$$dim(\mathcal{A}) \leq dim(M) - \frac{1}{2} Rk\{\,,\}.$$

Definition

If $(M, \{\,,\})$ is an affine Poisson variety whose algebra of Casimirs is maximal and $\mathcal{A}$ is a complete involutive subalgebra of $\mathcal{F}(M)$ then $\mathcal{A}$ is called *integrable* if

$$dim(\mathcal{A}) = dim(M) - \frac{1}{2} Rk\{\,,\}.$$

Given s functions $f_1, \dots, f_s \in \mathcal{F}(M)$, we denote by $F = (f_1, \dots, f_s)$ the s-tuple of this data.

Proposition

Let $(M, \{\,,\})$ be a Poisson manifold and let us suppose that $F = (f_1, \dots, f_s)$ is involutive. Then,

(a) *The Hamiltonian vector fields $X_{f_1}, \dots, X_{f_s}$ commute.*

(b) *The subalgebra of $\mathcal{F}(M)$, generated by the functions $f_1, \dots, f_s$ is also involutive.*

Definition

Let $(M, \{\,,\})$ be a Poisson manifold of rank $2r$ and set a s-tuple $F = (f_1, \dots, f_s)$ of elements in $\mathcal{F}(M)$. We say that F is *completely integrable*, in the sense of Liouville, if it is involutive, independent and $s = dim(M) - r$.
In this case, $(M, \{\,,\}, F)$ is said to be a *completely integrable system*.

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In this case, $(M, \{\,,\}, F)$ is said to be a *completely integrable system.*

Important result: *Liouville Theorem.*

Merci beaucoup à tous!

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